Math 60 7.8 Models Involving Rational Equations (Word Problems)

Objectives:

1) Ratio and proportion
   a) Ratio is a fraction. Proportion is an equation setting two ratios equal
   b) Check set-up
   c) Solve by cross-multiply

2) Conversions can be solved using proportions
   a) One fraction is the known conversion rate
   b) One fraction contains the known quantity and the unknown desired

3) Similar triangles can be solved by proportions
   a) Identify corresponding sides and corresponding angles
   b) Write ratios of corresponding sides

4) Work rates
   a) Assume everyone works at a constant rate no matter the circumstance
   b) Fraction of work done in one hour = $\frac{1}{total\ time}$
   c) Add fractions alone = fraction together

5) Uniform Motion
   a) $D = RT$ used when distance given [Math 45, chapter 2]
      i) Equal distances: $R_1T_1 = R_2T_2$
      ii) Sum of distances: $R_1T_1 + R_2T_2 = sum$ (traveling in opposite directions)
      iii) Difference of times: $R_1T_1 - R_2T_2 = difference$ (traveling in same direction)

   b) $\frac{D}{R} = T$ (divide formula by R) is used when time given
      i) Equal times: $\frac{D_1}{R_1} = \frac{D_2}{R_2}$
      ii) Sum of times: $\frac{D_1}{R_1} + \frac{D_2}{R_2} = sum$
      iii) Difference of times: $\frac{D_1}{R_1} - \frac{D_2}{R_2} = difference$

   c) $\frac{D}{T} = R$ (divide formula by T) is used when rate given
      i) Equal rates: $\frac{D_1}{T_1} = \frac{D_2}{T_2}$
      ii) Difference of rates: $\frac{D_1}{T_1} - \frac{D_2}{T_2} = difference$

Examples and Practice

1) Solve $\frac{n}{8} = \frac{56}{64}$

2) Laura spend 86 € (euros) on a vase. $1 US equals 0.75013 €. To the nearest cent, how much did the vase cost in US dollars?

3) A 1300-square-foot house in a particular neighborhood sold for $310,000. For how much should the agent price a 1700-square-foot house in the same neighborhood? Round to the nearest thousand dollars.

4) ΔABC is similar to ΔDEF, with AB=35, AC=14, DF=10 and DE=x. Find x.

5) A student is finding the height of a tree, by measuring the tree's shadow, then immediately measuring the shadow of a yardstick. The tree's shadow is 24 feet and the shadow of the yardstick is 2 feet. Find the height of the tree.

6) Dan, an experienced horticulturalist, can plant 1800 seedlings in 4 hours and Hala, a student assistant, takes 6 hours to plan 1800 seedlings. To the nearest tenth of an hour, how long would it take Dan and Hala working together to plant 1800 seedlings?

7) Bob can paint a 1500-square-foot apartment in 20 hours. If his son Mark, a college student, helps him, they can paint a 1500-square foot apartment in 12 hours. How long does it take Mark to paint the apartment alone?

8) Barbara is canoeing in a river with a 1 mph current. After traveling 12 miles downstream, she turns around and goes upstream 10 miles in the same amount of time it took her to go downstream. How fast does Barbara paddle in still water?

9) Every weekend, you ride your bicycle on a forest path 15 miles long, ending at a waterfall. Your cycling speed downhill from the waterfall is 2 miles per hour faster than your cycling speed uphill. If the round trip takes 8 hours, find the average cycling speed coming back from the waterfall.

Math 60 Examples and Practice 7.8

① Solve $\frac{n}{8} = \frac{56}{64}$

Method 1: cross-multiply $\frac{n}{8} \times \frac{56}{64}$

$n \cdot 64 = 8 \cdot 56$

$64n = 448$ simplify

$\frac{64n}{64} = \frac{448}{64}$ isolate n

$\boxed{n = 7}$

Method 2: multiply by LCD 8, 64 → LCD = 64

$64 \cdot \frac{n}{8} = 64 \cdot \frac{56}{64}$ mult by LCD both sides

$\frac{8n}{8} = \frac{56}{8}$ cancel

$\boxed{n = 7}$ isolate n.

Method 3: Notice that the fraction $\frac{56}{64}$ reduces

$\frac{n}{8} = \frac{56 \div 8}{64 \div 8}$ reduce fraction

$\frac{n}{8} = \frac{7}{8}$ mult by 8

$\boxed{n = 7}$

② 86€ for vase, \$1 US = 0.75013 €

$$\frac{86\ €}{x\ \$US} = \frac{0.75013\ €}{1\ \$US}$$

← both numerators are euros €

← both denominators are \$US.

↑ vase (entire fraction)

↑ conversion (entire fraction)

Check set up vertically and horizontally

Math 60 7.8

Statements on this page are true for any proportion.

NOTE: There are several correct versions of this proportion.

$$\frac{86\text{€}}{\$x} = \frac{0.75013\text{€}}{\$1}$$

$$\frac{\$x}{86\text{€}} = \frac{\$1}{0.75013\text{€}}$$

$$\frac{86\text{€}}{0.75013\text{€}} = \frac{\$x}{\$1}$$

$$\frac{0.75013\text{€}}{86\text{€}} = \frac{\$1}{\$x}$$

These versions still check vertically and horizontally;

All four will give the same correct answer.

NOTE: There are several WRONG proportions, too.

NOT $\frac{86\text{€}}{\$x} = \frac{\$1}{0.75013\text{€}}$ ← numerators have different units + diff items ☹ ← denominators have diff. units + diff items ☹

↑ both vase ☺ ↑ both conversion ☺

One fraction is upside-down.

NOT $\frac{86\text{€}}{\$1} = \frac{\$x}{0.75013\text{€}}$ ← numerators both vase ☺ ← denominators both conversion ☺

↑ different units + diff items ☹ ↑ different units + diff items ☹

One fraction is upside-down

NOT $\frac{\$x}{86\text{€}} = \frac{0.75013\text{€}}{\$1}$

NOT $\frac{\$1}{86\text{€}} = \frac{0.75013\text{€}}{\$x}$

All four of these wrong proportions will give the same wrong answer.

M60 7.8

$\frac{86}{x} = \frac{0.75013}{1}$ after checking, remove units

$86 = 0.75013x$ cross multiply

$\frac{86}{0.75013} = x$ isolate x

$x = 114.6467946$ calculator

$\boxed{\$114.65 \text{ US}}$ round to nearest hundredth to get dollars and cents.

Always include units!

Method 2: mult by LCD = x

$x \cdot \frac{86}{x} = \frac{0.75013}{1} \cdot x$

$86 = 0.75013x$ same as Method 1.

③ 1300 sq ft $310,000 ⇒ 1700 sq. ft x dollars

$\frac{1300 \text{ sq.ft}}{\$310,000} = \frac{1700 \text{ sq ft}}{\$x}$ ← same units ✓ ← same units ✓

↑ 1st house ✓ ↑ 2nd house ✓

$1300x = (1700)(310,000)$ cross multiply

$\frac{1300x}{1300} = \frac{527000000}{1300}$ isolate x

$x = 405384.6154$ calculator

$\boxed{x = \$405000}$ Round to nearest thousand.

Note: You could ÷100 first to remove some zeros:
$13x = 17(310,000)$
$x = \frac{5270000}{13}$

M60 7.8

Similar Triangles

$\triangle ABC$ is similar to $\triangle DEF$.

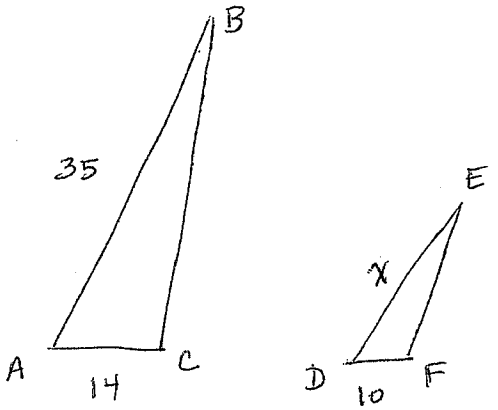
④ $\triangle ABC \sim \triangle DEF$ means the two triangles are the same shape but different sizes. Specifically, the angles can be paired so that

$\measuredangle A \cong \measuredangle D$

$\measuredangle B \cong \measuredangle E$

$\measuredangle C \cong \measuredangle F$

because of the order the vertices are listed when we write $\triangle ABC \sim \triangle DEF$.



Similarly, the corresponding sides are proportional.

$$\frac{AB}{DE} = \frac{AC}{DF}$$

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

We cannot calculate using two = signs, so we pick only two ratios and one = sign.

$$\frac{35}{x} = \frac{14}{10}$$

Substitute known values and unknown variable

$$35 \cdot 10 = 14x$$

cross-multiply

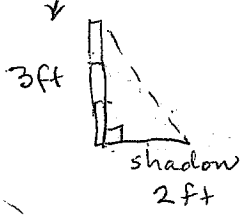
$$\frac{350}{14} = \frac{14x}{14}$$

isolate x

$$25 = x$$

⑤

Yardstick = 1 yard = 3 feet



Shadow problems are about similar triangles because the sun makes all the angles correspond.

M60 7.8

$\frac{x}{24} = \frac{3}{2}$ ← vertical sides of Δs ✓ ← bottoms of Δs ✓

↑ tree ✓ ↑ yardstick ✓ write and check proportion

$2x = 3 \cdot 24$ cross-multiply

$\frac{2x}{2} = \frac{72}{2}$ isolate x

$\boxed{x = 36 \text{ ft}}$

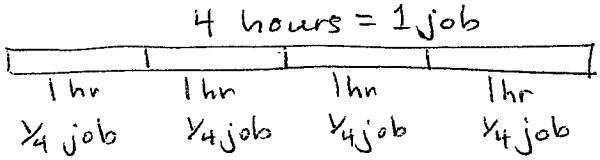
⑥ WORK RATES: These problems always refer to the same job (or equivalent jobs) being done by different combinations of workers.

IMPORTANT: Any number in the problem which described what or how big the job is can be ignored!

Our job is planting 1800 seedlings. We can ignore the 1800 and consider it one job.

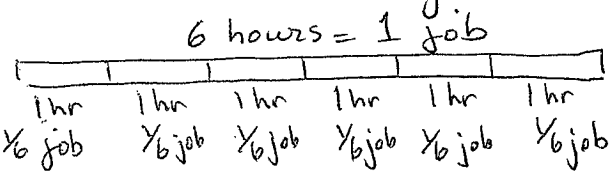
Dan does the job in 4 hours
Hala does the job in 6 hours
How long to do the job together?

What fraction of the job does Dan do in 1 hour? $\frac{1}{4}$.



Dan is a perfect worker, always working at a constant rate!

What fraction of the job does Hala do in 1 hour?



Hala also works at a constant rate, but it's slower

M60 7.8.

How much time to do the job together? x hours

What fraction of the job do they do in 1 hour? $\frac{1}{x}$ hours

Should x be greater than 6, between 4 and 6, or less than 4?

Working together = working faster (less time) than either person alone.

The answer x should be less than 4. (and less than 6).

Let's write our equation by thinking about ONE HOUR.

$\frac{1}{4}$ + $\frac{1}{6}$ = result is the fraction of the job they do together in 1 hour. = $\frac{1}{x}$

$\frac{1}{4}$: in 1 hour, Dan does $\frac{1}{4}$ of job

$\frac{1}{6}$: in 1 hour Hala does $\frac{1}{6}$ of job

They work together, so we add what they do.

Memorize the set up!

fraction alone + fraction alone = fraction together

Just the algebra:

$$\frac{1}{4} + \frac{1}{6} = \frac{1}{x}$$

Multiply by x:

$$\frac{x}{4} + \frac{x}{6} = 1$$

Some people memorize this phrase to help them remember this:

"Together-over-alone plus together-over-alone equals 1"

Either set-up is correct!

x = together
4 = Dan alone

x = together
6 = Hala alone

· m60 7.8

Solve $\frac{1}{4} + \frac{1}{6} = \frac{1}{x}$ find LCD = 12x

$12x \cdot \frac{1}{4} + 12x \cdot \frac{1}{6} = 12x \cdot \frac{1}{x}$ mult all terms by LCD.

$3x + 2x = 12$ cancel.

$5x = 12$ combine like terms

$x = \frac{12}{5}$ isolate x.

$x = 2.4$ hours calculate (no rounding needed this time)

Solve $\frac{x}{4} + \frac{x}{6} = 1$ find LCD = 12

$12 \cdot \frac{x}{4} + 12 \cdot \frac{x}{6} = 12 \cdot 1$ mult all terms by LCD * including RHS *

$3x + 2x = 12$ cancel

⋮ same as before.

⑦ 1500 sq-feet = 1 job. Ignore 1500.

Bob in 20 hours (alone) fraction $\frac{1}{20}$

Together 12 hours (together) fraction $\frac{1}{12}$

Mark in x hours (alone) fraction $\frac{1}{x}$

Fractions:

Add the "alone" fractions to get the "together" fraction

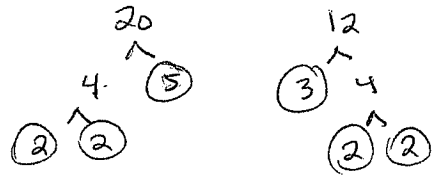
$\frac{1}{20} + \frac{1}{x} = \frac{1}{12}$

Together-over-alone plus together-over-alone = 1 gives

$\frac{12}{20} + \frac{12}{x} = 1$

M60 7.8

Solve $\frac{1}{20} + \frac{1}{x} = \frac{1}{12}$



$20 = 2 \cdot 2 \cdot 5$ $\quad 12 = 2 \cdot 2 \cdot 3$

$LCD = 2 \cdot 2 \cdot 3 \cdot 5 \cdot x = 60x$

$60x \cdot \frac{1}{20} + 60x \cdot \frac{1}{x} = 60x \cdot \frac{1}{12}$ mult all by LCD

$3x + 60 = 5x$ cancel

$60 = 2x$ collect x by $-3x$ both sides

$30 = x$ hrs — isolate x

Solve $\frac{12}{20} + \frac{12}{x} = 1$ notice $\frac{12}{20}$ reduces!

$\frac{12 \div 4}{20 \div 4} = \frac{3}{5}$

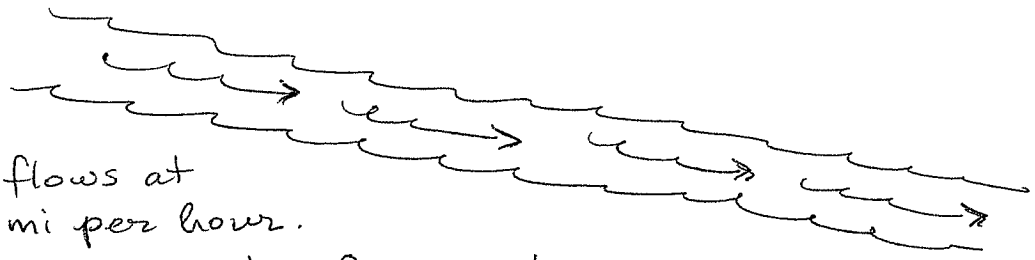
$\frac{3}{5} + \frac{12}{x} = 1$

$5x \cdot \frac{3}{5} + 5x \cdot \frac{12}{x} = 5x \cdot 1$ $\quad LCD = 5x$

$3x + 60 = 5x$ cancel

⋮ same as before

(8)



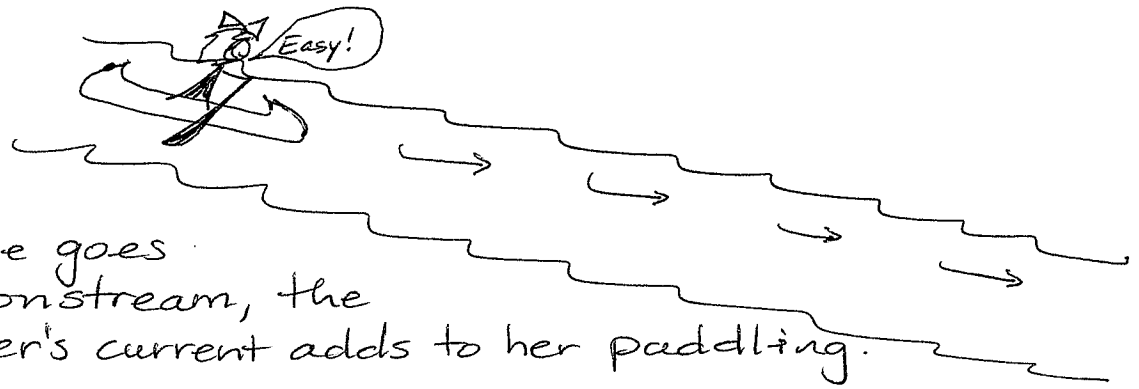
River flows at 1 mi per hour.

This is a rate! If we put a paper boat in the water, it will travel 1 mile each hour.

This is Barbara. In a completely still lake, she will go nowhere, unless she paddles. We don't know how fast she can paddle, so we call it x, also a rate, also miles per hour.

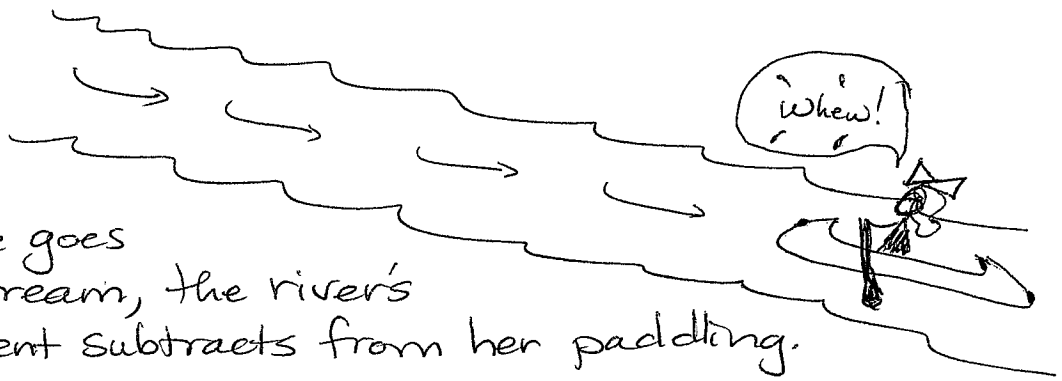
m60 7.8

When we put Barbara and her canoe on the river, it depends on which direction she goes.



If she goes downstream, the river's current adds to her paddling.

She goes $x+1$ miles per hour.



If she goes upstream, the river's current subtracts from her paddling.

She goes $x-1$ miles per hour.

Make a table

$D = R \cdot T$

T part of the table is blank at first.

Solve $D=RT$ for T by dividing both sides by R

$T = \frac{D}{R}$

	D	R	T
downstream	12	$x+1$	$\frac{D}{R} \rightarrow \frac{12}{x+1}$
upstream	10	$x-1$	$\frac{D}{R} \rightarrow \frac{10}{x-1}$

same means equal

we don't know T, so don't use a version of formula that has T.

$\frac{D=RT}{R \quad R}$

$\frac{D}{R} = T$

Time downstream = Time Upstream.

$\frac{12}{x+1} = \frac{10}{x-1}$

· M60 7.8

Solve $\frac{12}{x+1} = \frac{10}{x-1}$ proportion: cross-multiply

$12(x-1) = 10(x+1)$

$12x - 12 = 10x + 10$ dist $12(x-1)$, dist $10(x+1)$

$2x = 22$ subtract $10x$, add 12

$x = \frac{22}{2}$ isolate x.

$x = 11$ miles per hour

⑨ option 1 $D = R \cdot T$

	D	R	$\frac{D}{R} \rightarrow \frac{15}{x}$
up to fall	15	x	
down from fall	15	$x+2$	$\frac{D}{R} \rightarrow \frac{15}{x+2}$

In this table x = rate going to waterfall. It's a correct set-up, not the answer to the question!

add for total time = 8 hours

$\frac{15}{x} + \frac{15}{x+2} = 8$

Option 2: $D = R \cdot T$

	D	R	$\frac{D}{R} \rightarrow \frac{15}{x-2}$
up to fall	15	$x-2$	
down from fall	15	x	$\frac{D}{R} \rightarrow \frac{15}{x}$

add for total time = 8 hours

$\frac{15}{x-2} + \frac{15}{x} = 8$

m60 7.8

Option 1: Solve $\frac{15}{x} + \frac{15}{x+2} = 8$

then find $x+2$ to answer question.

$x(x+2)\cdot\frac{15}{x} + x(x+2)\cdot\frac{15}{x+2} = 8\cdot x(x+2)$ LCD = $x(x+2)$ mult all terms

$15(x+2) + 15x = 8x(x+2)$ cancel

$15x + 30 + 15x = 8x^2 + 16x$ dist $15(x+2)$ dist $8x(x+2)$

---- WOAH! WAKE UP!. x^2 means quadratic! ---

$30x + 30 = 8x^2 + 16x$ combine

$\frac{0}{2} = \frac{8x^2}{2} - \frac{14x}{2} - \frac{30}{2}$ set = 0.

$0 = 4x^2 - 7x - 15$ divide all by 2

$0 = (4x+5)(x-3)$

4(-15)
-60
-12 5
-7

Factor

$4x^2 - 7x - 15$
$4x^2 - 12x + 5x - 15$
$4x(x-3) + 5(x-3)$
$(4x+5)(x-3)$

$4x+5=0$ or $x-3=0$

$4x = -5$

$x = \frac{-5}{4}$ negative bicycling rate is extraneous

$x = 3$ mph going uphill

$x+2 = 2+3 =$ **5 mph down**

Option 2: Solve $\frac{15}{x-2} + \frac{15}{x} = 8$

$\frac{15}{x-2} \cdot x(x-2) + \frac{15}{x} \cdot x(x-2) = 8x(x-2)$ LCD = $x(x-2)$ mult all terms

$15x + 15(x-2) = 8x(x-2)$ cancel

$15x + 15x - 30 = 8x^2 - 16x$ dist $15(x-2)$ dist $8x(x-2)$

$30x - 30 = 8x^2 - 16x$ $-30x$ both sides $+30$ both sides

— — — — — WAKE UP — QUADRATIC x^2! — — —

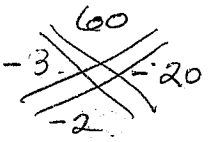
Because this is an equation, we can divide all by 2. →

$\frac{0}{2} = \frac{8x^2}{2} - \frac{4x}{2} + \frac{30}{2}$ set = 0

$0 = 4x^2 - 23x + 15$ divide by 2

$0 = (4x-3)(x-5)$

Factor



$4x^2 - 3x - 20x + 15$

$x(4x-3) - 5(4x-3)$

$(4x-3)(x-5)$

1, 60
2, 30
3, 20

$4x - 3 = 0$ $x - 5 = 0$

$4x = 3$ $x = 5$ mph

$x \neq 3/4$

Why is this extraneous?
Because the other factor
$x - 2 \Rightarrow 3/4 - 2 =$ negative!
doesn't make sense